

SOME IDENTITIES OF SYMMETRY FOR THE DEGENERATE q -BERNOULLI POLYNOMIALS UNDER SYMMETRY GROUP OF DEGREE n

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ABSTRACT. Recently, Kim-Kim Introduced some interesting identities of symmetry for q -Bernoulli polynomials under symmetry group of degree n . In this paper, we study the degenerate q -Euler polynomials and derive some identities of symmetry for these polynomials arising from the p -adic q -integral on \mathbb{Z}_p .

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . The p -adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $q \in \mathbb{C}_p$ be an indeterminate such that $|1 - q|_p < p^{-\frac{1}{p-1}}$. The q -analogue of the number x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Let $f(x)$ be Uniformly differentiable function on \mathbb{Z}_p . The p -adic q -integral on \mathbb{Z}_p is defined by Kim to be

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_q(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [13]}). \end{aligned} \tag{1.1}$$

In [1], L. Carlitz considered q -analogue of Bernoulli numbers which are given by recurrence relation to be

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{1.2}$$

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with the usual convention about replacing β_q^n by $\beta_{n,q}$. He defined q -Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l}, \quad (\text{see [1, 13]}). \quad (1.3)$$

In [13], Kim proved that the Carlitz's q -Bernoulli polynomials are represented as the p -adic q -integral on \mathbb{Z}_p which are given by

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0). \quad (1.4)$$

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are the Carlitz q -Bernoulli numbers.

In [2], L. Carlitz also introduced the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}^*(x) \frac{t^n}{n!}. \quad (1.5)$$

Note that $\lim_{\lambda \rightarrow 0} B_{n,\lambda}^*(x) = B_n^*(x)$, where $B_n(x)$ are ordinary Bernoulli polynomials (see [1-10]). When $x = 0$, $B_{n,\lambda}^* = B_{n,\lambda}^*(0)$ are called the degenerate Bernoulli numbers. Recently, Kim-Kim introduced (fully) degenerate Bernoulli polynomials which are derived from the p -adic invariant integral on \mathbb{Z}_p as follows:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_q(x) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}} \quad (\text{see [7]}), \quad (1.6)$$

where $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, and

$$\lim_{q \rightarrow 1} \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x).$$

The (fully) degenerate Bernoulli polynomials are defined by the generating function to be

$$\frac{\log(1+\lambda t)}{\lambda(1+\lambda t)^{\frac{1}{\lambda}} - \lambda} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} \quad (\text{see [7]}). \quad (1.7)$$

Note that Kim's degenerate Bernoulli polynomials are slightly different from the Carlitz's degenerate Bernoulli polynomials.

From (1.6) and (1.7), we note that

$$\lambda \int_{\mathbb{Z}_p} \left(\frac{x+y}{\lambda} \right)_n d\mu_1(x) = B_{n,\lambda(x)} \quad (n \geq 0), \quad (1.8)$$

where $(x)_0 = 1$, $(x)_n = x(x-1) \cdots (x-n+1)$, $(n \geq 1)$.

In [16], Kim considered degenerate q -Bernoulli polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda}[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,\lambda,q}(x) \frac{t^n}{n!}. \quad (1.9)$$

When $x = 0$, $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$ are called (fully) degenerate q -Bernoulli numbers. Note that $\lim_{\lambda \rightarrow 0} \beta_{n,\lambda,q}(x) = \beta_{n,q}(x)$, ($n \geq 0$).

In this paper, we give some identities of symmetry for the degenerate q -Bernoulli polynomials under symmetry group of degree n arising from the p -adic q -integral on \mathbb{Z}_p .

2. Identities of symmetry for the degenerate q -Bernoulli polynomials

We assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$. In this section, let w_1, w_2, \dots, w_n be positive integers. For $N \in \mathbb{N}$, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda}[w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} (\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i) k_j]_q} d\mu_q^{w_1 w_2 \cdots w_{n-1}}(y) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[w_n p^N]_{q^{w_1 \cdots w_{n-1}}}} \\ & \quad \times \sum_{y=0}^{w_n p^N - 1} (1 + \lambda t)^{\frac{1}{\lambda}[w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} (\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i) k_j]_q} q^{w_1 w_2 \cdots w_{n-1} y} \\ &= \lim_{N \rightarrow \infty} \frac{1}{[w_n p^N]_{q^{w_1 \cdots w_{n-1}}}} \sum_{k_n=0}^{w_n-1} \sum_{y=0}^{p^N-1} q^{w_1 w_2 \cdots w_{n-1} (k_n + w_n y)} \\ & \quad \times (1 + \lambda t)^{\frac{1}{\lambda}[(\sum_{j=1}^{n-1} w_j)(k_n + w_n y) + \sum_{j=1}^n w_j x + w_n \sum_{j=1}^{n-1} (\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i) k_j]_q}. \end{aligned} \quad (2.1)$$

From (2.1), we note that

$$\begin{aligned}
& \frac{1}{[w_1 \cdots w_{n-1}]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} q^{w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \\
& \times \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda} \left[w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q} d\mu_q^{w_1 w_2 \cdots w_{n-1}}(y) \\
& = \lim_{N \rightarrow \infty} \frac{1}{[w_1 \cdots w_n p^N]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} \sum_{k_n=0}^{p^N-1} \sum_{y=0}^{w_n-1} q^{w_1 w_2 \cdots w_{n-1} (k_n + w_n y) + \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j w_n} \\
& \times (1 + \lambda t)^{\frac{1}{\lambda} \left[\left(\sum_{j=1}^{n-1} w_j \right) (k_n + w_n y) + \sum_{j=1}^{n-1} w_j x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q}.
\end{aligned} \tag{2.2}$$

It is easy to show that (2.2) is invariant under any permutation in the symmetry group of degree n . Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1. *Let w_1, w_2, \dots, w_n be positive integers. Then, the following expressions*

$$\begin{aligned}
& \frac{1}{[w_{\sigma(1)} \cdots w_{\sigma(n-1)}]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q^{w_{\sigma(n)} \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_{\sigma(i)} \right) k_j} \\
& \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda} \left[w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)} y + \sum_{j=1}^n w_j x + w_{\sigma(n)} \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q} d\mu_q^{w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}}(y)
\end{aligned}$$

are the same for any permutation σ in the symmetry group of order n .

It is not difficult to show that

$$\begin{aligned}
& \left[w_1 w_2 \cdots w_{n-1} y + w_1 w_2 \cdots w_n x + w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q \\
& = [w_1 w_2 \cdots w_{n-1}]_q \left[y + w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right]_{q^{w_1 w_2 \cdots w_{n-1}}}.
\end{aligned} \tag{2.3}$$

From (2.3), we note that

$$\begin{aligned}
& \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda} [w_1 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1} (\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i) k_j]}_q d\mu_q^{w_1 w_2 \cdots w_{n-1}}(y) \\
&= \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{[w_1 \cdots w_{n-1}]_q}{\lambda} \left[y + w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right]}_q d\mu_q^{w_1 \cdots w_{n-1}}(y) \\
&= \int_{\mathbb{Z}_p} \left(1 + \frac{\lambda}{[w_1 \cdots w_{n-1}]_q} [w_1 \cdots w_{n-1}]_q t \right)^{\frac{[w_1 \cdots w_{n-1}]_q}{\lambda} \left[y + w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right]}_q d\mu_q^{w_1 \cdots w_{n-1}}(y) \\
&= \sum_{m=0}^{\infty} [w_1 \cdots w_{n-1}]_q^m \beta_{m, \frac{\lambda}{[w_1 \cdots w_{n-1}]_q}, q}^{w_1 \cdots w_{n-1}} \left(w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

Therefore, by Theorem 1 and (2.4), we obtain the following theorem.

Theorem 2.2. *For $m \geq 0$, $w_1, w_2, \dots, w_n \in \mathbb{N}$, the following expressions*

$$\begin{aligned}
& [w_{\sigma(1)} \cdots w_{\sigma(n-1)}]_q^{m-1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_{\sigma(l)}-1} q^{\sum_{j=1}^{n-1} (\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_{\sigma(i)}) k_j w_{\sigma(n)}} \\
& \times \beta_{m, \frac{\lambda}{[w_{\sigma(1)} \cdots w_{\sigma(n-1)}]_q}, q}^{w_{\sigma(1)} \cdots w_{\sigma(n-1)}} \left(w_{\sigma(n)} x + \frac{w_{\sigma(n)}}{w_{\sigma(1)}} k_1 + \cdots + \frac{w_{\sigma(n)}}{w_{\sigma(n-1)}} k_{n-1} \right)
\end{aligned}$$

are the same for any permutation σ in the symmetry group of order n .

From (1.9), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n, \lambda, q}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda} [x+y]_q} d\mu_q(y) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda} \right) d\mu_q(x) \lambda^n t^n \\
&= \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda} \right)_n d\mu_q(x) \frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

By comparing the coefficients on the both sides of (2.5), we get

$$\begin{aligned}
\beta_{n,\lambda,q} &= \lambda^n \int_{\mathbb{Z}_p} \left(\frac{[x+y]_q}{\lambda} \right)_n d\mu_q(x) \\
&= \lambda^n \sum_{m=0}^n S_1(n, m) \lambda^{-m} \int_{\mathbb{Z}_p} [x+y]_q^m d\mu_q(y) \\
&= \sum_{m=0}^n S_1(n, m) \lambda^{n-m} \beta_{m,q}(x).
\end{aligned} \tag{2.6}$$

where $\beta_{m,q}(x)$ are called Carlitz's q -Bernoulli polynomials.

Now, we observe that

$$\begin{aligned}
&\left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_{q^{w_1 \cdots w_{n-1}}} \\
&= \frac{[w_n]_q}{[w_1 \cdots w_{n-1}]_q} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}} + q^{w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} [y + w_n x]_{q^{w_1 \cdots w_{n-1}}}.
\end{aligned} \tag{2.7}$$

By (2.6), we get

$$\begin{aligned}
&\beta_{m, \frac{\lambda}{[w_1 \cdots w_{n-1}]_q}, q^{w_1 \cdots w_{n-1}}} \left(w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right) \\
&= \left(\frac{\lambda}{[w_1 \cdots w_{n-1}]_q} \right)^m \int_{\mathbb{Z}_p} \left(\left(\frac{\lambda}{[w_1 \cdots w_{n-1}]_q} \right)^{-1} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_q^l \right) d\mu_{q^{w_1 \cdots w_{n-1}}}(y) \\
&= \left(\frac{\lambda}{[w_1 \cdots w_{n-1}]_q} \right)^m \sum_{l=0}^m S_1(m, l) [w_1 \cdots w_{n-1}]_q^l \lambda^{-l} \\
&\quad \times \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_q^l d\mu_{q^{w_1 \cdots w_{n-1}}}(y).
\end{aligned} \tag{2.8}$$

From (2.7), we can derive the following equation:

$$\begin{aligned}
& \int_{\mathbb{Z}_p} \left[y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right]_q^l d\mu_{q^{w_1 \cdots w_{n-1}}} (y) \\
&= \sum_{s=0}^l \binom{l}{s} \left(\frac{[w_n]_q}{[w_1 \cdots w_{n-1}]_q} \right)^{l-s} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{l-s} q^{w_n s \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \\
&\quad \times \int_{\mathbb{Z}_p} [y + w_n x]_{q^{w_1 \cdots w_{n-1}}}^s d\mu_{q^{w_1 \cdots w_{n-1}}} (y) \\
&= \sum_{s=0}^l \binom{l}{s} \left(\frac{[w_n]_q}{[w_1 \cdots w_{n-1}]_q} \right)^{l-s} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{l-s} q^{w_n s \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \\
&\quad \times \beta_{s,q^{w_1 \cdots w_{n-1}}} (w_n x).
\end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we get

$$\begin{aligned}
& \beta_{m, \frac{\lambda}{[w_1 \cdots w_{n-1}]_q}, q^{w_1 \cdots w_{n-1}}} \left(w_n x + \frac{w_n}{w_1} k_1 + \cdots + \frac{w_n}{w_{n-1}} k_{n-1} \right) \\
&= \sum_{p=0}^m \sum_{s=0}^p \binom{p}{s} S_1(m, p) \lambda^{m-p} [w_1 \cdots w_{n-1}]_q^{s-m} [w_n]_q^{p-s} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{p-s} \\
&\quad \times q^{w_n s \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \beta_{s,q^{w_1 \cdots w_{n-1}}} (w_n x).
\end{aligned} \tag{2.10}$$

From (2.10), we note that

$$\begin{aligned}
& [w_1 \cdots w_{n-1}]_q^{m-1} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} q^{w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \\
& \times \beta_{m, \frac{\lambda}{[w_1 \cdots w_{n-1}]_q}, q^{w_1 \cdots w_{n-1}}} \left(w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j} \right) \\
& = \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} \sum_{p=0}^m \sum_{s=0}^p \binom{p}{s} S_1(m, p) \lambda^{m-p} [w_1 \cdots w_{n-1}]_q^{s-1} [w_n]_q^{p-s} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{p-s} \\
& \quad \times q^{(s+1)w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \beta_{s, q^{w_1 \cdots w_{n-1}}} (w_n x) \\
& = \sum_{p=0}^m \sum_{s=0}^p \binom{p}{s} S_1(m, p) \lambda^{m-p} [w_1 \cdots w_{n-1}]_q^{s-1} [w_n]_q^{p-s} \beta_{s, q^{w_1 \cdots w_{n-1}}} (w_n x) \\
& \quad \times \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} q^{(s+1)w_n \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_{q^{w_n}}^{p-s} \\
& = \sum_{p=0}^m \sum_{s=0}^p \binom{p}{s} S_1(m, p) \lambda^{m-p} [w_1 \cdots w_{n-1}]_q^{s-1} [w_n]_q^{p-s} \beta_{s, q^{w_1 \cdots w_{n-1}}} (w_n x) \\
& \quad \times K_{n, q^{w_n}}(w_1, \cdots, w_{n-1} | p-s, s), \tag{2.11}
\end{aligned}$$

where

$$K_{n, q}(w_1, \cdots, w_{n-1} | i, t) = \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} q^{(t+1) \sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j} \left[\sum_{j=1}^{n-1} \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} w_i \right) k_j \right]_q^i. \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.3. *Let $m \geq 0$ and $w_1, w_2, \cdots, w_n \in \mathbb{N}$, Then the following expressions*

$$\begin{aligned}
& \sum_{p=0}^m \sum_{s=0}^p \binom{p}{s} S_1(m, p) \lambda^{m-p} [w_{\sigma(1)} \cdots w_{\sigma(n-1)}]_q^{s-1} [w_{\sigma(n)}]_q^{p-s} \beta_{s, q^{w_{\sigma(1)} \cdots w_{\sigma(n-1)}}} (w_{\sigma(n)} x) \\
& \times K_{n, q^{w_{\sigma(n)}}} (w_{\sigma(1)}, \cdots, w_{\sigma(n-1)} | p-s, s)
\end{aligned}$$

are the same for any permutation σ in the symmetry group of order n .

Note that some identities of Bernoulli and Euler polynomials are studied by several authors (see [1-19]).

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